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On the final sequence of a finitary set functor

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Abstract

A standard construction of the final coalgebra of an endofunctor involves defining a chain of iterates, starting at the final object of the underlying category and successively applying the functor. In this paper we show that, for a finitary set functor, this construction always yields a final coalgebra in $\omega 2 = \omega + \omega$ steps.

1. Introduction

The theorems of Aczel and Mendler [2], and Barr [10], guarantee the existence of final coalgebras for a wide class of endofunctors on Set. In each case one can, in theory, derive a general recipe for constructing final coalgebras from the proof. However, in practice it is difficult to learn much about the structure of the final coalgebra of a specific endofunctor. Aczel and Mendler obtain a final coalgebra as a quotient (by bisimilarity) of a coproduct of a set of coalgebras. Barr shows that if a set functor T is *accessible* (cf. Section 2) then the category of T-coalgebras has a set of generators. He then uses the Special Adjoint Functor Theorem, whose proof also involves a quotient-of-a-sum construction, to derive the existence of a final coalgebra. Work on the problem of providing more concrete constructions of final coalgebras includes the set-theoretic representations of Aczel [1] and Paulson [15],

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the *coalgebraic logic* of Moss [14], and a domain representation of non-well-founded sets in Mislove et al. [13].

In this paper we adopt the approach of Adámek and Koubek [5] (and of Barr in another paper [9]). We consider an endofunctor T on a category $\mathcal{C}$ with limits of ordinal-indexed diagrams, and define the *final sequence* of T: an ordinal-indexed sequence $\langle A_\alpha \rangle$ of objects of $\mathcal{C}$, with arrows $f^\alpha_\beta : A_\alpha \to A_\beta$ for $\beta \leqslant \alpha$. Briefly, this is defined by $A_{\alpha+1} = TA_\alpha$, and $A_\lambda = \mathrm{Lim}_{\alpha<\lambda} A_\alpha$ for λ a limit ordinal. Fuller details are given in the next section. It is shown in [5] that if this sequence stabilizes at some α, in the sense that $f^{\alpha+1}_\alpha$ is an isomorphism, then $(A_\alpha, (f^{\alpha+1}_\alpha)^{-1})$ is a final T-coalgebra. This generalizes the iterative construction of the greatest fixed point of a monotone function f on a complete lattice as the stabilizing value of the ordinal-indexed sequence $\langle a_\alpha \rangle$, where $a_\alpha = \sqcap_{\beta<\alpha} f(a_\beta)$.

For set functors, accessibility seems to be a common denominator amongst some of the hypotheses involved in the various final coalgebra theorems in the literature, e.g., being *bounded* in [12] and *set-based* in [2]. Adámek and Porst [6] have shown that the assumption of boundedness of a set functor is equivalent to accessibility. Aczel and Mendler actually consider set based endofunctors on the category of classes and class functions, but, as Barr shows, this basically amounts to assuming an inaccessible cardinal[2] κ and considering κ-accessible set functors preserving the subcategory of sets of size less than or equal to κ. Thus we are led to study the final sequences of accessible set functors.

Adámek and Koubek [5] show that for set functors the mere existence of a final coalgebra is sufficient to ensure stabilization of the final sequence. They do not, in general, give bounds for stabilization, although in the case of the finite powerset functor $\mathbb{P}$ they show stabilization at ω_1—the first uncountable ordinal. The main point we wish to make in this paper is that for finitary (ω-accessible) endofunctors on Set the construction of the final coalgebra via the final sequence is a two-stage process, each of which is finitary. More precisely, the final sequence stabilizes in $\omega 2 = \omega + \omega$ steps. A corresponding result holds if we replace ω with any regular cardinal κ.

The first stage of the final-coalgebra construction can be seen as taking a Cauchy completion of the initial algebra, while the next stage can be seen as pruning this to obtain the final coalgebra. We show that, in general, the ωth iterate in the final sequence of a finitary set functor is always a final coalgebra—not necessarily of T, but certainly of the lifting of T to an endofunctor on the category of complete ultrametric spaces and nonexpansive maps. We give two examples to support these intuitions. In particular, for the finite powerset functor $\mathbb{P}$, the first ω steps in the final sequence construct the set of *compactly branching*, strongly extensional trees. In the next ω steps of the final sequence, these trees are pruned, one level at a time, until we reach the set of *finitely branching*, strongly extensional trees.

We assume that the reader is acquainted with the notions of category, functor, limits and colimits. Otherwise the paper is self-contained. This work is based on the conference paper [21].

[2] A cardinal κ is inaccessible if $\lambda < \kappa$ implies that $2^\lambda < \kappa$. The fact that we talk about accessibility for cardinals on the one hand, and for functors on the other, is purely coincidental!

2. Coalgebras and final sequences

In this section we recall the notions of a coalgebra of an endofunctor and the final sequence of an endofunctor. The latter is defined in [9,5], while Rutten [16] is a good introduction to the theory and applications of coalgebras. We also recall the notion of an accessible functor [7].

2.1. Coalgebras

A *coalgebra* of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ is a pair $(A, f : A \to TA)$, where A, the carrier of the coalgebra, is an object of $\mathcal{C}$, and f, the structure map, is an arrow of $\mathcal{C}$. A *homomorphism* of T-coalgebras (A, f) and (B, g) is an arrow $h : A \to B$ such that the diagram below commutes in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & TA \arrow[d, "Th"] \\ B \arrow[r, "g"'] & TB \end{tikzcd} \tag{1}$$

This definition gives a category of T-coalgebras and T-coalgebra homomorphisms. A final object of this category, if it exists, is called a final T-coalgebra. In this paper we only consider the case where T is an endofunctor on Set.

Next we introduce a condition on a set functor T to ensure the existence of a final T-coalgebra. A cardinal κ is *regular* if it is not the sum of fewer than κ strictly smaller cardinals. For example, ω and ω_1 are regular. For a regular cardinal κ we say that a partially ordered set I is κ-*directed* if each subset of I with size strictly less than κ has an upper bound in I. A functor $T : \mathsf{Set} \to \mathsf{Set}$ is κ-*accessible* if it preserves colimits of those diagrams indexed by κ-directed posets.[3] An ω-accessible functor is sometimes called *finitary*.

Example 1. Our leading example of a finitary set functor is the finite powerset functor $\mathbb{P} : \mathsf{Set} \to \mathsf{Set}$. For a set X, $\mathbb{P}X$ is the collection of finite subsets of X. For a function $f : X \to Y$, $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is defined by $(\mathbb{P}f)(S) = f(S)$.

Another finitary set functor that we will consider is the finite subprobability distributions functor $\mathbb{D} : \mathsf{Set} \to \mathsf{Set}$. For a set X, $\mathbb{D}X$ is the set of functions $\mu : X \to [0, 1]$ such that $\mu(x) > 0$ for at most finitely many $x \in X$ and $\sum_{x \in X} \mu(x) \leqslant 1$. For $\mu \in \mathbb{D}X$ and $E \subseteq X$ define $\mu[E] = \sum_{x \in E} \mu(x)$. We can extend $\mathbb{D}$ to an endofunctor on Set by defining, for a function $f : X \to Y$, $(\mathbb{D}f)(\mu)(y) = \mu[f^{-1}(y)]$. The functor $\mathbb{D}$ was studied by De Vink and Rutten [20] in connection with the notion of probabilistic bisimulation.

2.2. Final sequences

Let $\mathcal{C}$ be a category with limits of all ordinal-indexed cochains, and T an endofunctor on $\mathcal{C}$. The *final sequence* of T is an ordinal-indexed sequence of objects $\langle A_\alpha \rangle$, with maps

[3] Note that the preservation of κ-directed colimits is equivalent to the preservation of all κ-filtered colimits, since for any κ-filtered category $\mathcal{A}$ there is a κ-directed poset I and a cofinal functor $F : I \to \mathcal{A}$, cf. [7].

$(f_\gamma^\beta : A_\beta \to A_\gamma)_{\gamma \leqslant \beta}$, uniquely defined by the following conditions (where $\delta \leqslant \gamma \leqslant \beta$):

- FS-1 $A_{\beta+1} = T(A_\beta)$,
- FS-2 $f_{\gamma+1}^{\beta+1} = T(f_\gamma^\beta)$,
- FS-3 $f_\beta^\beta = \mathrm{id}$,
- FS-4 $f_\delta^\beta = f_\delta^\gamma \cdot f_\gamma^\beta$,
- FS-5 if β is a limit ordinal, the cone $(f_\gamma^\beta : A_\beta \to A_\gamma)_{\gamma \leqslant \beta}$ is a limit.

We define the sequence by ordinal induction, checking at each stage that conditions [FS-1]–[FS-5] hold for the portion of the sequence already defined.

(1) *Case*: α is a limit ordinal. We define $(f_\beta^\alpha : A_\alpha \to A_\beta)_{\beta<\alpha}$ to be the limit of the cochain $\langle A_\beta \rangle_{\beta<\alpha}$, and we set $f_\alpha^\alpha = \mathrm{id}$. Conditions [FS-1]–[FS-5] are easily verified.

(2) *Case*: $\alpha = \alpha'+1$. We define $A_\alpha = T(A_{\alpha'})$ and $f_\alpha^\alpha = \mathrm{id}$. Next we define the projections f_β^α by induction on $\beta < \alpha$. If $\beta < \alpha$ is a successor ordinal, say $\beta = \beta' + 1$, then we define $f_\beta^\alpha = T(f_{\beta'}^{\alpha'})$. If β is a limit ordinal, and if the maps f_γ^α have already been defined for all $\gamma < \beta$, then by the universal property of A_β there is a unique map f_β^α making [FS-4] true.

Theorem 2 (*Adámek and Koubek [5], Barr [10]*). *Suppose the final sequence of T stabilizes at κ, in the sense that $f_\kappa^{\kappa+1}$ is an isomorphism, then $(A_\kappa, (f_\kappa^{\kappa+1})^{-1})$ is a final T-coalgebra.*

3. Final sequences of set functors

This section contains our main result, Theorem 11, stating that for a regular cardinal κ the final sequence of a κ-accessible set functor stabilizes in $\kappa 2$ steps.

Example 3. To motivate the general development, we consider the final sequence $\{A_\alpha, f_\beta^\alpha\}$ of the finite powerset functor $\mathbb{P}\colon \mathsf{Set} \to \mathsf{Set}$. Both [5] and [19] show that, even though this functor is finitary, the final coalgebra cannot be constructed by the usual ω^{op}-limit, i.e., the final sequence does not stop in ω steps. In fact, the projection $f_\omega^{\omega+1}$ is not surjective, since any sequence $\langle B_i \rangle$ in the image of $f_\omega^{\omega+1}$ is *uniformly bounded* in the sense that there exists N such that $|B_i| \leqslant N$ for all i. To see this, suppose that $\langle B_i \rangle = f_\omega^{\omega+1}(Y)$ for some $Y \in A_{\omega+1}$; then

$$B_{n+1} = f_{n+1}^{\omega}(\langle B_i \rangle) = (f_{n+1}^{\omega} \cdot f_\omega^{\omega+1})(Y) = (\mathbb{P} f_n^\omega)(Y).$$

Setting $B_{i+1} = A_i \in A_{i+1}$ defines a sequence $\langle B_i \rangle$ in A_ω that is not uniformly bounded.

However $f_\omega^{\omega+1}$ is easily seen to be injective. Let $S = \{d_1, \ldots, d_l\} \subseteq A_\omega$, $T = \{e_1, \ldots, e_m\} \subseteq A_\omega$ and suppose $f_\omega^{\omega+1}(S) = f_\omega^{\omega+1}(T)$. Then

$$\begin{aligned}(\mathbb{P} f_n^\omega)(S) &= f_{n+1}^{\omega+1}(S) = f_{n+1}^{\omega}(f_\omega^{\omega+1}(S)) = f_{n+1}^{\omega}(f_\omega^{\omega+1}(T)) = f_{n+1}^{\omega+1}(T)\\ &= (\mathbb{P} f_n^\omega)(T)\end{aligned}$$

for all $n < \omega$. Now pick $d_i \in S$. Since T is finite there exists $e_j \in T$ such that $f_n^\omega(d_i) = f_n^\omega(e_j)$ for infinitely many n; thus $d_i = e_j$. This proves that $S \subseteq T$ and the converse follows by symmetry.

We will revisit this example in Section 5. Next we generalize the observation in the last part of the example with the following lemma.

Lemma 4. *Let* $T : \mathsf{Set} \to \mathsf{Set}$ *be* κ*-accessible for some regular cardinal* κ*. Given a* κ^{op}*-cochain*

$$A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_\alpha \leftarrow \cdots,$$

with nonempty limit, then the natural connecting map $\mathrm{Lim}_{\alpha<\kappa} TA_\alpha \leftarrow T(\mathrm{Lim}_{\alpha<\kappa} A_\alpha)$ *is injective.*

Proof. Suppose $(L \xrightarrow{p_\alpha} A_\alpha)_{\alpha<\kappa}$ and $(L' \xrightarrow{q_\alpha} TA_\alpha)_{\alpha<\kappa}$ are limiting cones, with $f : TL \to L'$ the connecting map. L is nonempty by assumption; thus L is the κ-directed colimit of all its nonempty subsets of size less than κ. Write $(S_i \xrightarrow{c_i} L)_{i\in I}$ for the colimiting cocone.

For each $i \in I$, since the image of c_i has cardinality less than κ, and since κ is regular, there exists $\alpha < \kappa$ such that $p_\alpha \cdot c_i$ is injective. Since the domain of c_i is nonempty, $p_\alpha \cdot c_i$ is a split mono. It follows that $T(p_\alpha \cdot c_i)$ is an injective map. But

$$\begin{aligned} T(p_\alpha \cdot c_i) &= Tp_\alpha \cdot Tc_i \\ &= q_\alpha \cdot f \cdot Tc_i. \end{aligned}$$

So $f \cdot Tc_i$ is injective for each $i \in I$.

Since $(TS_i \xrightarrow{Tc_i} TL)_{i\in I}$ is a directed colimit, any two elements of TL are in the image of Tc_i for some $i \in I$. It follows that f is injective. □

From now on we consider the final sequence $\{A_\alpha, f^\alpha_\beta\}$ of a κ-accessible endofunctor T on Set.

Proposition 5. *Let* (E, e) *be a T-coalgebra. We can extend* (E, e) *to a cone* $(e_\alpha : E \to A_\alpha)_\alpha$ *over the final T-sequence such that* $e_{\alpha+1} = Te_\alpha \cdot e$*.*

Proof. We define the maps e_α by transfinite induction, verifying at each stage that we have $(\forall \beta \leqslant \alpha) f^\alpha_\beta \cdot e_\alpha = e_\beta$. The successor clause is $e_{\alpha+1} \overset{\mathrm{def}}{=} Te_\alpha \cdot e$. Then

$$\begin{aligned} f^{\alpha+1}_\beta \cdot e_{\alpha+1} &= f^{\beta+1}_\beta \cdot f^{\alpha+1}_{\beta+1} \cdot Te_\alpha \cdot e \\ &= f^{\beta+1}_\beta \cdot T(f^\alpha_\beta \cdot e_\alpha) \cdot e \\ &= f^{\beta+1}_\beta \cdot Te_\beta \cdot e \\ &= f^{\beta+1}_\beta \cdot e_{\beta+1} \\ &= e_\beta. \end{aligned}$$

For a limit ordinal λ, we define e_λ by $f^\lambda_\beta \cdot e_\lambda = e_\beta$ for all $\beta < \lambda$. □

Lemma 6. *T is either the constant* $\emptyset$*, or every set* A_α *in the final sequence is nonempty.*

Proof. Suppose $T \neq \emptyset$. Then there exists $E \neq \emptyset$ with $TE \neq \emptyset$. Thus there exists a T-coalgebra (E, e), and this may be extended to a cone over the final T-sequence as in Proposition 5. It follows that $A_\alpha \neq \emptyset$ for all α. $\square$

Without loss of generality we assume that T is not the constant $\emptyset$ functor. Then combining Lemmas 4 and 6 we conclude that $f_\kappa^{\kappa+1}$ must be a split mono. Moreover, since T preserves split monos, and the projections from the limit of a cochain of injections are themselves injective, a simple induction establishes that f_β^α is injective for all $\alpha \geqslant \beta \geqslant \kappa$.

Many set functors preserve arbitrary intersections, that is, they preserve wide pullbacks of monos. For such functors (an example is $\mathbb{P}$), once we know that $f_\kappa^{\kappa+1}$ is injective it follows that the final sequence stabilizes in $\kappa + \omega$ steps—since $A_{\kappa+\omega}$ is just the intersection of the 'decreasing' cochain $\langle A_{\kappa+n} \rangle_{n<\omega}$. It is known that all set functors preserve finite nonempty intersections, cf. Trnková [18], however there are set functors that do not preserve infinite nonempty intersections, cf. Gumm [11]. Nevertheless, we are able to prove that the final sequence of any ω-accessible set functor stabilizes in $\omega 2$ steps. The proof uses the fact that $f_\omega^{\omega+1}$ is split mono, and thus yields a coalgebra structure on A_ω. This coalgebra turns out to be weakly final, and we obtain a final coalgebra as a retract. More generally, the final sequence of any κ-accessible set functor stabilizes in $\kappa 2$ steps.

Since $f_\kappa^{\kappa+1}$ is an injection with nonempty domain we may choose $l \colon A_\kappa \to A_{\kappa+1}$ such that $l \cdot f_\kappa^{\kappa+1} = \mathrm{id}$.

Proposition 7. *The T-coalgebra (A_κ, l) is weakly final.*

Proof. Let (E, e) be a T-coalgebra. (E, e) extends to a cone $(e_\alpha : E \to A_\alpha)_\alpha$ over the final sequence of T such that $e_\kappa = f_\kappa^{\kappa+1} \cdot e_{\kappa+1} = f_\kappa^{\kappa+1} \cdot Te_\kappa \cdot e$. It follows that $l \cdot e_\kappa = Te_\kappa \cdot e$, that is, e_κ is a T-coalgebra morphism from (E, e) to (A_κ, l). $\square$

Proposition 8. *Extend the coalgebra (A_κ, l) to a cone $(l_\alpha \colon A_\kappa \to A_\alpha)_\alpha$ over the final sequence of T. Then l_κ is an idempotent map of coalgebras $(A_\kappa, l) \to (A_\kappa, l)$.*

Proof. That l_κ is a coalgebra homomorphism follows from the proof of Proposition 7. Next we prove by induction on $\alpha \leqslant \kappa$ that $l_\alpha \cdot l_\kappa = l_\alpha$. The successor case is:

$$\begin{aligned} l_{\alpha+1} \cdot l_\kappa &= Tl_\alpha \cdot l \cdot l_\kappa \\ &= Tl_\alpha \cdot Tl_\kappa \cdot l \qquad (l_\kappa \text{ a coalgebra map}) \\ &= T(l_\alpha \cdot l_\kappa) \cdot l \\ &= Tl_\alpha \cdot l \\ &= l_{\alpha+1}. \end{aligned}$$

Case: α a limit ordinal. For all $\beta < \alpha$ we have that $f_\beta^\alpha \cdot l_\alpha \cdot l_\kappa = l_\beta \cdot l_\kappa = l_\beta = f_\beta^\alpha \cdot l_\alpha$; thus $l_\alpha \cdot l_\kappa = l_\alpha$. $\square$

All idempotents in Set split, so we can write $l_\kappa = i \cdot q$, where $q : A_\kappa \to G$ and $i : G \to A_\kappa$ satisfy $q \cdot i = \mathrm{id}$. Notice that the pair $Tq \cdot l$ and $f_\kappa^{\kappa+1} \cdot Ti$ is also a splitting of l_κ, thus, by the uniqueness of splittings of idempotents, we have an isomorphism g making the diagram below commute.

$$\begin{tikzcd} A_\kappa \arrow[r, "l"] \arrow[d, "q"'] & A_{\kappa+1} \arrow[d, "Tq"] \\ G \arrow[r, "g"] \arrow[d, "i"'] & TG \arrow[d, "Ti"] \\ A_\kappa & A_{\kappa+1} \arrow[l, "f_\kappa^{\kappa+1}"] \end{tikzcd} \tag{2}$$

Proposition 9. *The T-coalgebra (G, g) is final.*

Proof. Clearly (G, g) is weakly final since (A_κ, l) is weakly final. Suppose h and k are two coalgebra maps $(E, e) \to (G, g)$. (E, e) extends to a cone $(e_\alpha : E \to A_\alpha)_\alpha$ over the final T-sequence; we show by induction that $f_\alpha^\kappa \cdot i \cdot h = e_\alpha$ for all $\alpha < \kappa$. The case for α a limit ordinal is trivial. The successor case is:

$$\begin{aligned} f_{\alpha+1}^\kappa \cdot i \cdot h &= f_{\alpha+1}^\kappa \cdot i \cdot g^{-1} \cdot Th \cdot e \qquad (h \text{ a coalgebra map}) \\ &= f_{\alpha+1}^\kappa \cdot f_\kappa^{\kappa+1} \cdot Ti \cdot Th \cdot e \qquad (\text{cf. diagram } (2)) \\ &= f_{\alpha+1}^{\kappa+1} \cdot Ti \cdot Th \cdot e \\ &= T(f_\alpha^\kappa \cdot i \cdot h) \cdot e \\ &= Te_\alpha \cdot e \\ &= e_{\alpha+1}. \end{aligned}$$

Similarly one proves that $f_\alpha^\kappa \cdot i \cdot k = e_\alpha$ for all $\alpha < \kappa$. It follows that $i \cdot h = i \cdot k$, and hence that $h = k$. □

Proposition 10. *Let $l_\kappa = i \cdot q$ be as in Diagram* (2). *Then*
(i) $l_\kappa \cdot f_\kappa^{\kappa 2} = f_\kappa^{\kappa 2}$,
(ii) $q \cdot f_\kappa^{\kappa 2}$ *is injective.*

Proof. We show by induction on $\alpha \leqslant \kappa$ that

$$(\forall \alpha \leqslant \kappa) \quad l_\alpha \cdot f_\kappa^{\kappa+\alpha} = f_\alpha^{\kappa+\alpha}.$$

Case: α a limit ordinal. For all $\beta < \alpha$ it holds that

$$\begin{aligned} f_\beta^\alpha \cdot l_\alpha \cdot f_\kappa^{\kappa+\alpha} &= l_\beta \cdot f_\kappa^{\kappa+\alpha} \\ &= l_\beta \cdot f_\kappa^{\kappa+\beta} \cdot f_{\kappa+\beta}^{\kappa+\alpha} \\ &= f_\beta^{\kappa+\beta} \cdot f_{\kappa+\beta}^{\kappa+\alpha} \\ &= f_\beta^{\kappa+\alpha} \\ &= f_\beta^\alpha \cdot f_\alpha^{\kappa+\alpha}. \end{aligned}$$

Thus $l_\alpha \cdot f_\kappa^{\kappa+\alpha} = f_\alpha^{\kappa+\alpha}$.

Case: α a successor ordinal.

$$\begin{aligned}
l_{\alpha+1} \cdot f_{\kappa}^{\kappa+\alpha+1} &= T l_\alpha \cdot l \cdot f_{\kappa}^{\kappa+\alpha+1}\\
&= T l_\alpha \cdot l \cdot f_{\kappa}^{\kappa+1} \cdot f_{\kappa+1}^{\kappa+\alpha+1}\\
&= T l_\alpha \cdot f_{\kappa+1}^{\kappa+\alpha+1}\\
&= T(l_\alpha \cdot f_{\kappa}^{\kappa+\alpha})\\
&= T f_{\alpha}^{\kappa+\alpha}\\
&= f_{\alpha+1}^{\kappa+\alpha+1}.
\end{aligned}$$

This completes the proof of (i).

For (ii) observe that $l_\kappa \cdot f_\kappa^{\kappa 2} = i \cdot q \cdot f_\kappa^{\kappa 2}$ is injective by part (i). A fortiori $q \cdot f_\kappa^{\kappa 2}$ is injective. $\square$

Theorem 11. *If T is a κ-accessible endofunctor on* Set *with final sequence* $\{A_\alpha, f_\beta^\alpha\}$, *then* $f_{\kappa 2}^{\kappa 2+1}$ *is an isomorphism.*

Proof. The final T-coalgebra (G, g), as constructed in Proposition 9, extends to a cone $(g_\alpha : G \to A_\alpha)_\alpha$ over the final T-sequence. By definition of this cone, the top square in diagram (3) commutes. The bottom square of this diagram is just the top square of (2), thus it also commutes. The middle square commutes by definition of the final T-sequence. Thus the whole diagram commutes.

The map $q \cdot f_\kappa^{\kappa 2}$ is injective by Proposition 10. By the finality of (G, g) we have that $q \cdot f_\kappa^{\kappa 2} \cdot g_{\kappa 2}$ is the identity, so $q \cdot f_\kappa^{\kappa 2}$ is also surjective, and hence an isomorphism of sets. It follows that three sides of the lower rectangle are isomorphisms, thus the top side $f_{\kappa 2}^{\kappa 2+1}$ is also an isomorphism. $\square$

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "g_{\kappa 2}"'] & TG \arrow[d, "T g_{\kappa 2}"] \\
A_{\kappa 2} \arrow[d, "f_\kappa^{\kappa 2}"'] & TA_{\kappa 2} \arrow[l] \arrow[d, "T f_\kappa^{\kappa 2}"] \\
A_\kappa \arrow[d, "q"'] & TA_\kappa \arrow[l] \arrow[d, "Tq"] \\
G \arrow[r, "g"] & TG
\end{tikzcd} \tag{3}$$

Theorem 11 is restricted to set functors because we use the fact that in Set injectives with nonempty domain are split mono. In fact, once one has that $f_\omega^{\omega+1}$ is a split mono, to conclude that $f_{\omega 2}^{\omega 2+1}$ is an isomorphism one only needs to know that idempotents split in the underlying category.

We can easily extend Theorem 11 to functors that may be neither κ-accessible, nor κ^{op}-continuous. An example, in the case that $\kappa = \omega$, is the functor $\mathbb{P}(-)^A$ where A is infinite. The coalgebras of this functor are the so-called *image-finite* transition systems (those transition systems such that for each state s, and label $a \in A$, the set of states reachable from s by a one-step a-labelled transition is finite). First we recall the following simple proposition which says that coproducts commute with κ^{op}-limits in Set.

Proposition 12. *Suppose we have a family of κ^{op}-limit cochains in* Set

$$X_{i0} \leftarrow X_{i1} \leftarrow \cdots \leftarrow X_{i\alpha} \leftarrow X_{i(\alpha+1)} \leftarrow \cdots \leftarrow X_{i\kappa}$$

indexed over $i \in I$; then

$$\coprod X_{i0} \leftarrow \coprod X_{i1} \leftarrow \cdots \leftarrow \coprod X_{i\alpha} \leftarrow \coprod X_{i(\alpha+1)} \leftarrow \cdots \leftarrow \coprod X_{i\kappa}$$

is also a limit cochain for each i.

Theorem 13. *The class of endofunctors on* Set *such that the arrow $f_\kappa^{\kappa+1}$ in their final sequence is injective is closed under*:
(1) *κ-accessible functors,*
(2) *κ^{op}-continuous functors,*
(3) *composition of functors,*
(4) *arbitrary coproducts of functors,*
(5) *$\mathcal{I}$-indexed-limits of functors, for any small category $\mathcal{I}$.*

Proof. Consider the following property of an endofunctor T: for all κ^{op}-limits

$$B_0 \leftarrow B_1 \leftarrow \cdots \leftarrow B_\alpha \leftarrow \cdots \leftarrow B_\kappa,$$

with B_κ nonempty, the connecting map $\mathrm{Lim}_{\alpha<\kappa}\, TB_\alpha \leftarrow TB_\kappa$ is injective. We prove that this property is closed under (1)–(5) above.

Closure under 1 was shown in Lemma 4, while closure under 2 is trivial. Closure under 4 holds by Proposition 12, and the fact that in Set coproducts preserve monos. For 3 suppose $S, T : \mathsf{Set} \to \mathsf{Set}$, and assume the property in question holds of S and T. Either T is the constant $\emptyset$ functor, in which case $S \cdot T$ is constant, or $TX \neq \emptyset$ for $X \neq \emptyset$, in which case the composition

$$\operatorname*{Lim}_{\alpha<\kappa} STB_\alpha \leftarrow S\left(\operatorname*{Lim}_{\alpha<\kappa} TB_\alpha\right) \leftarrow STB_\kappa$$

is injective since S preserves injections with nonempty domain. Finally, for closure under 5, suppose $T = \mathrm{Lim}_{I\in\mathcal{I}}\, T_I$. By assumption, for each $I \in \mathcal{I}$, the connecting map $\mathrm{Lim}_{\alpha<\kappa}\, T_I B_\alpha \leftarrow T_I B_\kappa$ is injective. Recall that limits commute with each other in a complete category and that the functor $\mathrm{Lim}_{I\in\mathcal{I}}(-) : [\mathcal{I}, \mathsf{Set}] \to \mathsf{Set}$, being a right adjoint, preserves injections. Thus the connecting map $\mathrm{Lim}_{\alpha<\kappa}\, TB_\alpha \leftarrow TB_\kappa$, which is the composite

$$\operatorname*{Lim}_{\alpha} TB_\alpha \cong \operatorname*{Lim}_{\alpha} \operatorname*{Lim}_{I} T_I B_\alpha \cong \operatorname*{Lim}_{I} \operatorname*{Lim}_{\alpha} T_I B_\alpha \longleftarrow \operatorname*{Lim}_{I} T_I B_\kappa \cong TB_\kappa$$

is injective.

The theorem now holds since an endofunctor on Set is either the constant $\emptyset$, or each set in its final sequence is nonempty (cf. the remark following Lemma 4). □

Corollary 14. *The class of endofunctors on* Set *whose final sequences stabilize in at most* $\kappa 2$ *steps is closed under* (1)–(5) *in Theorem* 13.

Proof. This follows from the proof of Theorem 11. $\square$

Example 15. We consider a slight variant of the functor $\mathbb{D}$ introduced in Example 1. That is, we let $\mathbb{D}_c X$ be the set of countably supported subprobability distributions on the set X. $\mathbb{D}_c$ is not finitary, and does not immediately appear to be covered by Corollary 14 above. However it is still the case that the final sequence $\{A_\alpha, f^\alpha_\beta\}$ of $\mathbb{D}_c$ stabilizes in $\omega 2$ steps. For suppose $\mu, \rho \in \mathbb{D}_c A_\omega$ and $f^{\omega+1}_\omega(\mu) = f^{\omega+1}_\omega(\rho)$. Then for each $\langle x_n \rangle \in A_\omega$, since $\mu[-]$ (being a measure) preserves decreasing countable intersections,

$$\mu(\langle x_n \rangle) = \mu\left[\bigcap_{n<\omega} (f^\omega_n)^{-1}(x_n)\right] = \lim_{n\to\infty} \mu[(f^\omega_n)^{-1}(x_n)] = \lim_{n\to\infty} (\mathbb{D}_c f^\omega_n)(\mu)(x_n).$$

Similarly we get $\rho(\langle x_n \rangle) = \lim_{n\to\infty} (\mathbb{D}_c f^\omega_n)(\rho)(x_n)$. But

$$\begin{aligned}(\mathbb{D}_c f^\omega_n)(\mu) &= f^{\omega+1}_{n+1}(\mu) = f^\omega_{n+1}(f^{\omega+1}_\omega(\mu)) = f^\omega_{n+1}(f^{\omega+1}_\omega(\rho)) \\ &= f^{\omega+1}_{n+1}(\rho)(\mathbb{D}_c f^\omega_n)(\rho)\end{aligned}$$

for all $n < \omega$. Thus $\mu = \rho$.

4. Set-theoretic versus metric final semantics

The idea of using final coalgebras of set functors to model infinite data types is due to Aczel and Mendler [2]. An alternative approach [8,20] is to consider final coalgebras of endofunctors on the category CUMet of complete ultrametric spaces and nonexpansive maps. In fact, there is no loss of generality in restricting attention to the full subcategory $\mathsf{CUMet}^\star$ of spaces where each pair of distinct points has distance 2^{-n} for some $n \in \mathbb{N}$. A comparison of the set-based and metric approaches turns out to be instructive in studying the final sequence of a set functor.

In this section we define a lifting of a set functor T to a locally contractive endofunctor $T^\star$ on $\mathsf{CUMet}^\star$. For example, the finite powerset functor $\mathbb{P}$ gives rise to the compact powerdomain functor $\mathcal{P}_{\mathrm{k}}$ (modulo a contraction factor). Furthermore, the ωth iterate in the final sequence of T is, in a natural ultrametric, a final coalgebra of $T^\star$. Thus, the final T-coalgebra can be seen as a subspace (indeed, a sub-T-algebra) of the final $T^\star$-coalgebra. For instance, in the next section the final coalgebra of $\mathcal{P}_{\mathrm{k}}((-)_{1/2})$ is described as the coalgebra of strongly extensional, *compactly branching* trees, and the final coalgebra of $\mathbb{P}$ is described as the coalgebra of strongly extensional, *finitely branching* trees.

Example 16. We start by recalling some relevant functors on $\mathsf{CUMet}^\star$.

(i) The compact powerdomain functor $\mathcal{P}_{\mathrm{k}} : \mathsf{CUMet}^\star \to \mathsf{CUMet}^\star$ maps a space $\langle X, d\rangle$ to the space $\mathcal{P}_{\mathrm{k}}\langle X, d\rangle$ of all compact subsets of X equipped with the *Hausdorff metric* d_{H}, where

$$d_{\mathrm{H}}(V, W) = \max\left\{\sup_{v\in V} \inf_{w\in W} d(v, w),\ \sup_{w\in W} \inf_{v\in V} d(v, w)\right\},$$

with all sups and infs taken over the interval [0, 1].

(ii) The scaling functor $(-)_{\frac{1}{2}}$ maps a space $\langle X, d\rangle$ to the space $\langle X, \frac{1}{2}d\rangle$ with the same set of points, but with all distances halved.

(iii) We say that a Borel measure μ on an ultrametric space $\langle X, d\rangle$ has *compact support* if there exists a compact set $K \subseteq X$ such that for all Borel sets U, $U \cap K = \emptyset$ implies $\mu(U) = 0$. Let $\mathcal{M}_{\mathrm{k}}\langle X, d\rangle$ denote the ultrametric space of Borel probability measures on $\langle X, d\rangle$ with compact support, where

$$d_{\mathcal{M}X}(\mu, \rho) = \inf\{\varepsilon > 0 \mid (\forall x \in X)\mu(B_\varepsilon(x)) = \rho(B_\varepsilon(x))\}$$

with $B_\varepsilon(x)$ the open ε-ball around $x \in X$.
If $\langle X, d\rangle$ is a complete ultrametric space then so is $\mathcal{M}_{\mathrm{k}}\langle X, d\rangle$. $\mathcal{M}_{\mathrm{k}}$ is turned into an endofunctor on $\mathsf{CUMet}^\star$ by defining $(\mathcal{M}_{\mathrm{k}} f)(\mu)(O) = \mu(f^{-1}(O))$ for a nonexpansive map $f : X \to Y$. See [20] for further details about this functor.

Given a set functor T, the definition of $T^\star$ is based on the well-known characterization of complete ultrametric spaces as pro-discrete objects in the category of topological spaces [17, Theorem 6.4.7]. Given an ultrametric space $\langle X, d\rangle$, for each $n \in \mathbb{N}$, the open balls $B_{2^{-n}}(x)$ form a partition of X. We denote this partition P_n. If $\langle X, d\rangle$ is complete, then the set X may be recovered as the limit of the ω^{op}-chain

$$P_0 \xleftarrow{g_0^1} P_1 \xleftarrow{g_1^2} P_2 \xleftarrow{g_2^3} \cdots, \tag{4}$$

where $g_n^m : P_m \to P_n$ is defined by $g_n^m(B_{2^{-m}}(x)) = B_{2^{-n}}(x)$ for $m \geqslant n$. The limit projection $g_n^\omega : X \to P_n$ is defined by $g_n^\omega(x) = B_{2^{-n}}(x)$.

Defintion 17. The endofunctor $T^\star$ on $\mathsf{CUMet}^\star$ is defined on objects by requiring that $T^\star\langle X, d\rangle$ has underlying set given by the limit (in Set) of the ω^{op}-chain

$$1 \longleftarrow TP_0 \xleftarrow{Tg_0^1} TP_1 \xleftarrow{Tg_1^2} TP_2 \xleftarrow{Tg_2^3} \cdots \tag{5}$$

obtained by applying T to (4) and appending 1, and distance function

$$d'(\langle x_n\rangle, \langle y_n\rangle) = \inf\{2^{-n} \mid x_n = y_n\}.$$

The functorial extension of $T^\star$ is straightforward once it is recalled that a nonexpansive map $f : \langle X, d_X\rangle \to \langle Y, d_Y\rangle$ yields a natural transformation from the decomposition (4) of X to the corresponding decomposition of Y.

Notice that in going from (4) to (5) the sequence of terms is shifted one place to the right. This has the effect of making $T^\star$ *locally contractive* [8], that is, its action on homsets defines a contractive function.

Let T have final sequence $\{A_\alpha, f^\alpha_\beta\}$. As observed by Barr [10], A_ω has a natural ultrametric given by

$$d_{A_\omega}(\langle x_n\rangle, \langle y_n\rangle) = \inf\{2^{-n} \mid x_n = y_n\}. \tag{6}$$

From the definition of $T^\star$ it immediately follows that $T^\star\langle A_\omega, d_{A_\omega}\rangle \simeq \langle A_\omega, d_{A_\omega}\rangle$. But America and Rutten [8] show that any fixed point of a locally contractive endofunctor of $\mathsf{CUMet}^\star$ is a final coalgebra. Thus we have:

Theorem 18. *$\langle A_\omega, d_{A_\omega}\rangle$ can be given the structure of a final $T^\star$-coalgebra.*

We devote the rest of this section to describing $T^\star$ in a couple of instances. For this it is worthwhile introducing a new characterization of $T^\star\langle X, d\rangle$ as the Cauchy completion of *TX* in a suitable metric.

Observe that there is a unique function $\iota: TX \to T^\star\langle X, d\rangle$ defined by $\pi_{n+1}\cdot\iota = T(g^\omega_n)$, where $(\pi_{n+1}: T^\star\langle X, d\rangle \to TP_n)_{n<\omega}$ is the limiting cone in Definition 17. If T is finitary, then ι is monic by Proposition 4. Furthermore, we have:

Proposition 19. *The image of ι is dense in $T^\star\langle X, d\rangle$.*

Proof. Since each map g^m_n in (4) is surjective, we can find maps $h^n_m: P_n \to P_m$ for $n \leqslant m$ and $h^n_\omega: P_n \to X$ satisfying

- G1 $g^{n+1}_n \cdot h^n_{n+1} = 1_{P_n}$,
- G2 $h^n_n = 1_{P_n}$,
- G3 $h^n_m = h^p_m \cdot h^n_p$ for $n \leqslant p \leqslant m$,
- G4 $g^\omega_m \cdot h^n_\omega = h^n_m$ for $n \leqslant m$.

(The maps h^n_m are uniquely determined by G2–G4 once each h^n_{n+1} is chosen to satisfy G1.)

Now, given a typical element $\langle x_n\rangle$ of $T^\star\langle X, d\rangle$, we have

$$(\pi_{m+1}\cdot\iota\cdot T(h^m_\omega))(x_{m+1}) = (T(g^m_\omega)\cdot T(h^m_\omega))(x_{m+1}) = x_{m+1}.$$

It immediately follows that the image of ι is dense in $T^\star\langle X, d\rangle$ as claimed. □

Regarding *TX* as a dense subset of $T^\star\langle X, d\rangle$, the restriction of the metric d' on $T^\star\langle X, d\rangle$ to *TX* is characterized by

$$d'(x, y) = \inf\{2^{-(n+1)} \mid T(g^\omega_n)(x) = T(g^\omega_n)(y)\}, \tag{7}$$

where the maps g^ω_n refer to (4). Recalling that $g^\omega_n(x) = g^\omega_n(y)$ iff $d(x, y) \leqslant 2^{-n}$, it is straightforward that, in case $T = \mathbb{P}$, (7) yields the Hausdorff metric on $\mathbb{P}X$ (modulo a contraction factor of $\frac{1}{2}$). Similarly, if $T = \mathbb{D}$, then (7) defines the metric of Example 16(iii) on probability distributions. We omit the details. Combining these observations with Proposition 20 below, we conclude that $\mathbb{P}^\star = \mathcal{P}_{\mathrm{k}}((-)_{\frac{1}{2}})$ and $\mathbb{D}^\star = \mathcal{M}_{\mathrm{k}}((-)_{\frac{1}{2}})$.

Proposition 20. *Let $\langle X, d\rangle$ be a complete ultrametric space. Then $\mathbb{P}X$ is a dense subset of $\mathcal{P}_{\mathrm{k}}\langle X, d\rangle$, and $\mathbb{D}X$ is a dense subset of $\mathcal{M}_{\mathrm{k}}\langle X, d\rangle$.*

Proof. Let $K \subseteq X$ be compact and let $\varepsilon > 0$. There is a finite set $K' = \{x_1, \ldots, x_m\} \subseteq K$ such that $\{B_\varepsilon(x) \mid x \in K'\}$ covers K. It is clear that the distance between K' and K in the Hausdorff metric is less than ε, proving the first assertion above. For the second, let μ be a Borel measure with compact support K above. Observe that without loss of generality we may assume that the $B_\varepsilon(x_i)$ are pairwise disjoint, since in an ultrametric space two open ε-balls are either disjoint or equal. Define the distribution ρ by $\mathrm{support}(\rho) = \{x_1, \ldots, x_m\}$ and $\rho(x_j) = \mu(B_\varepsilon(x_j))$ for $1 \leqslant j \leqslant m$. Then the distance between μ and ρ in $\mathcal{M}_{\mathrm{k}}X$ is less than ε since if $O \in \mathcal{O}_\varepsilon$ then each $B_\varepsilon(x_j)$ is either a subset of O or does not meet O, so

$$\mu(O) = \textstyle\sum_{x_j \in O} \mu(B_\varepsilon(x_j)) = \sum_{x_j \in O} \rho[B_\varepsilon(x_j)] = \rho(O). \qquad \square$$

5. The final sequence of $\mathbb{P}$

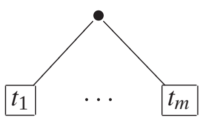
In this section we sketch an application of the final-coalgebra construction from the proof of Theorem 11 to the case of the finite powerset functor $\mathbb{P}$. This example illustrates well why one needs $\omega 2$ steps to ensure the stabilization of the final sequence of a finitary set functor.

For our purposes, a tree t is a directed graph with a distinguished node, the root, such that every node is reachable from the root by a unique finite path. We consider trees that are isomorphic as directed graphs with distinguished nodes to be identical. Given a node x of t, the *maximal subtree rooted at* x is the greatest subgraph of t that is a tree with root x.

A relation R on the set of nodes of a tree is a *tree bisimulation* if xRy implies the respective parents of x and y are related, each child of x is R-related to a child of y, and each child of y is R^{-1}-related to a child of x. We call a tree *strongly extensional* if no two distinct nodes are related by a bisimulation. (This notion basically goes back to the work of Aczel [1] on non-well-founded sets.) For any tree t the union of all tree bisimulations is an equivalence, and the quotient of t by this equivalence is strongly extensional. We write $t \approx_n t'$ if the restrictions of t and t' to depth n have the same strongly extensional quotient, and define a pseudo-metric $d_{\mathcal{T}}$ on the class of strongly extensional trees by

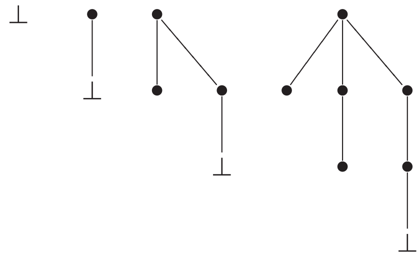
$$d_{\mathcal{T}}(t, t') = \inf\{2^{-n} \mid t \approx_n t'\}.$$

Write $\{\bot\}$ for the final object in Set, and $\{A_\alpha, f^\alpha_\beta\}$ for the final sequence of $\mathbb{P}$. For each $n < \omega$ there is an isomorphism between A_n and the set of finite-branching strongly extensional trees of depth not greater than n, and whose depth-n nodes are labelled $\bot$. This is defined by induction: if $x_1, \ldots, x_m \in A_n$ correspond to trees $t_1, \ldots, t_m$, then $\{x_1, \ldots, x_m\}$ corresponds to

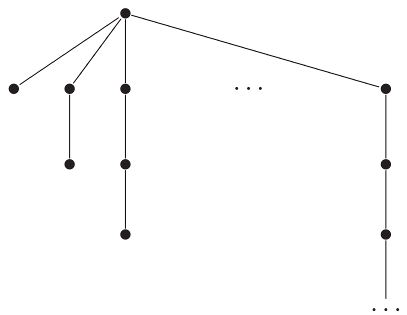


(8)

Labelling the leaf nodes of depth n in a tree in A_n by $\bot$ suggests that this tree is to be thought of as partial. We can think of the projection map f^{n+1}_n as taking a tree in A_{n+1}, cutting off the depth-$(n+1)$ nodes, taking the strongly extensional quotient, and finally relabelling the depth-n nodes by $\perp$. Below, we draw a row of trees taken respectively from A_0, A_1, A_2 and A_3, with each tree projecting down to the tree to the left.



Extending the given sequence in the obvious way, we can imagine the trees as projections of the following countably-branching infinite-depth tree in A_ω (this picture appears in Turi and Rutten [19]).



Notice that since the trees in A_ω are potentially infinitely branching, the obvious candidate for a coalgebra structure on A_ω, i.e., the map sending a tree t to the set of its maximal proper subtrees, does not work. However, in the last section we showed that A_ω can be given a coalgebra structure, indeed the structure of a final coalgebra, but of the compact powerdomain functor, not $\mathbb{P}$. In fact, the limit A_ω can be characterized as the set of strongly extensional trees that are compactly branching in the sense that, for each node, the set of maximal subtrees rooted at its children is compact with respect to the metric $d_{\mathcal{T}}$. The projection f^ω_n is given by: cut a tree to depth n, take the strongly extensional quotient, and relabel the depth-n nodes by $\perp$. The injection $f^{\omega+1}_\omega$ can be thought of as the inclusion of the subset of those trees that are finitely branching at the root. More generally, the injection $f^{\omega+n}_\omega$ can be thought of as the inclusion of the subset of those trees that are finitely branching up-to depth $n < \omega$. Finally, the carrier of the final coalgebra, $A_{\omega 2}$, is the set of finitely branching, strongly extensional trees. The coalgebra structure on this set sends each tree t to the set of its maximal proper subtrees.

6. Future work

The example in Section 5 shows that the $\omega 2$ bound on the stabilization of the final sequence of a finitary set functor is tight. However, we do not have an example of an ω_1-accessible set functor whose final sequence takes fully $\omega_1 + \omega_1$ steps to stabilize. As we mentioned earlier, if the functor preserves wide pullbacks of monos then one has stabilization in $\omega_1 + \omega$ steps. This case seems to cover all of the 'reasonable' ω_1-accessible set functors one comes across, e.g., the countable powerset functor.

Recently, Adámek [3] has posed the question of whether the behaviour one observes of the final sequence of a finitary set functor extends to finitary endofunctors on other locally finitely presentable categories. He obtains a partial answer to this question, showing that on certain locally finitely presentable categories, finitary endofunctors preserving strong monos and bimorphisms also have final sequences that stabilize in $\omega 2$ steps. The proof of this result uses Theorem 11. In the absence of any counter-examples, the question of whether one can drop any of the side conditions on the functor remains open.

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References

[1] P. Aczel, Non-well-founded sets, CLSI Lecture Notes, Vol. 14, Center for the Study of Languages and Information, Stanford, 1988.
[2] P. Aczel, P.F. Mendler, A final coalgebra theorem, in: Category Theory and Computer Science, Lecture Notes in Computer Science, Vol. 389, Springer, Berlin, 1989, pp. 357–365.
[3] J. Adámek, On final coalgebras of continuous functors, Theoret. Comput. Sci. 294 (2003) 3–29.
[4] J. Adámek, V. Koubek, Least fixed point of a functor, J. Comput. System Sci. 19 (1979) 163–178.
[5] J. Adámek, V. Koubek, On the greatest fixed point of a set functor, Theoret. Comput. Sci. 150 (1995) 57–75.
[6] J. Adámek, H.E. Porst, From varieties of algebras to covarieties of coalegbras, Proc. CMCS'01, Electron. Notes Theoret. Comput. Sci. 44 (2001).
[7] J. Adámek, J. Rosický, Locally presentable and accessible categories, LMS Lecture Notes, Vol. 189, Cambridge University Press, Cambridge, 1994.
[8] P. America, J. Rutten, Solving reflexive domain equations in a category of complete metric spaces, J. Comput. System Sci. 39 (3) (1989) 343–375.
[9] M. Barr, Algebraically compact functors, J. Pure Appl. Algebra 82 (1992) 211–231.
[10] M. Barr, Terminal coalgebras in well-founded set theory, Theoret. Comput. Sci. 114 (1993) 299–315.
[11] H.P. Gumm, Functors for coalgebras, Algebra Universalis 45 (2001) 135–147.
[12] Y. Kawahara, M. Mori, A small final coalgebra theorem, Theoret. Comput. Sci. 233 (1–2) (2000) 129–145.
[13] M.W. Mislove, L.S. Moss, F.J. Oles, Non-well-founded sets modelled as ideal fixed points, Inform. and Comput. 93 (1) (1991) 16–54.
[14] L.S. Moss, Coalgebraic logic, Ann. Pure Appl. Logic 99 (1–3) (1999) 241–259.
[15] L.C. Paulson, Final coalgebras as greatest fixed points in ZF set theory, Math. Struct. Comput. Sci. 9 (5) (1999) 545–567.
[16] J.J.M.M. Rutten, Universal coalgebra: a theory of systems, Theoret. Comput. Sci. 249 (1) (2000) 3–80.
[17] M.B. Smyth, Topology, in: S. Abramsky, D. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, Oxford University Press, Oxford, 1990.
[18] V. Trnková, Some properties of set functors, Comment. Math. Univ. Carolinae 10 (1969) 323–352.
[19] D. Turi, J. Rutten, On the foundations of final semantics: non-standard sets, metric spaces, partial orders, Math. Struct. Comput. Sci. 8 (5) (1998) 481–540.

[20] E. de Vink, J. Rutten, Bisimulation for probabilistic transition systems: a coalgebraic approach, Theoret. Comput. Sci. 221 (1–2) (1999) 271–293.

[21] J.B. Worrell, Terminal sequences for accessible endofunctors, Proceedings of Coalgebraic Methods in Computer Science ‘99, Elect. Notes Theor. Comp. Sci. 19 (1999) 39–54.